Chapter 3

Theory of Quadratic TFDs [0]

The quadratic time-frequency distributions (TFDs) introduced in the last chapter represent the majority of the methods used in practical applications that deal with non-stationary signals. In this chapter, which completes the introductory tutorial, we show that these particular quadratic TFDs belong to a general class of TFDs whose design follows a common procedure, and whose properties are governed by common laws. This quadratic class may be considered as the class of *smoothed* Wigner-Ville distributions (WVDs), where the "smoothing" is described in the (t, f) domain by convolution with a "time-frequency kernel" function $\gamma(t, f)$, and in other domains by multiplication and/or convolution with various transforms of $\gamma(t, f)$. The generalized approach allows the definition of new TFDs that are better adapted to particular signal types, using a simple and systematic procedure as opposed to the *ad hoc* methods of Chapter 2.

The first section (3.1) extends Section 2.1 by enumerating in detail the key properties and limitations of the WVD. Thus it motivates the introduction of general quadratic TFDs (Section 3.2) and prepares for the discussion of their properties (Section 3.3). In Section 3.2, using Fourier transforms from lag to frequency and from time to Doppler (frequency shift), the quadratic TFDs and their kernels are formulated in four different but related two-dimensional domains. One of these, namely the Doppler-lag (ν, τ) domain, leads to the definition of the "ambiguity function" and allows the smoothing of the WVD to be understood as a filtering operation. In Section 3.3, the list of properties of the WVD is supplemented by mentioning some desirable TFD properties not shared by the WVD. The various TFD properties are then expressed in terms of constraints on the kernel, so that TFD design is reduced to kernel design. Three tables are provided showing the kernel properties equivalent to various TFD properties, the kernels of numerous popular TFDs in the various two-dimensional domains, and the properties of those same TFDs.

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3.1 The WVD

3.1.1 Properties of the WVD

We have seen that the WVD has the simplest time-lag kernel (see Eq. (2.7.7)), and that each of the other TFDs can be written as a filtered WVD using a specific time-lag kernel filter (see Eq. (2.7.37)). In this sense the WVD is the basic or prototype TFD and the other TFDs are variations thereon. Moreover, we see in Fig. 2.7.1 that the WVD gives the sharpest indication of the IF law of a linear FM signal. These are some of the reasons why the WVD is the most widely studied TFD and deserves a further detailed description of its properties, as listed below.

- **Realness (RE):** $W_z(t, f)$ is real for all z, t and f.

- **Time-shift invariance** (also called **time covariance**): A time shift in the signal causes the same time shift in the WVD; that is, if

$$z_r(t) = z(t - t_0) \ , \tag{3.1.1}$$

  then

$$W_{z_r}(t, f) = W_z(t - t_0, f). \tag{3.1.2}$$

- **Frequency-shift invariance** (also called **frequency covariance**): A frequency shift in the signal causes the same frequency shift in the WVD; that is, if

$$z_r(t) = z(t)\, e^{j2\pi f_0 t} \ , \tag{3.1.3}$$

  then

$$W_{z_r}(t, f) = W_z(t, f - f_0). \tag{3.1.4}$$

- **Time marginal (TM):** Integration of the WVD over frequency gives the instantaneous power:

$$\int_{-\infty}^{\infty} W_z(t, f)\, df = |z(t)|^2. \tag{3.1.5}$$

- **Frequency marginal (FM):** Integration of the WVD over time gives the energy spectrum:

$$\int_{-\infty}^{\infty} W_z(t, f)\, dt = |Z(f)|^2. \tag{3.1.6}$$

- **Global energy:** Integration of the WVD over the entire (t, f) plane yields the signal energy E_z:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_z(t, f)\, dt\, df = E_z. \tag{3.1.7}$$

- **Instantaneous frequency (IF):** For an analytic signal, the first moment [i.e. the mean] of the WVD w.r.t. frequency is the IF:

$$\frac{\int_{-\infty}^{\infty} f\, W_z(t,f)\, df}{\int_{-\infty}^{\infty} W_z(t,f)\, df} = \frac{1}{2\pi}\frac{d}{dt}[\arg z(t)]\,. \tag{3.1.8}$$

- **Time delay (TD):** The first moment of the WVD w.r.t. time is the TD:

$$\frac{\int_{-\infty}^{\infty} t\, W_z(t,f)\, dt}{\int_{-\infty}^{\infty} W_z(t,f)\, dt} = -\frac{1}{2\pi}\frac{d}{df}[\arg Z(f)]\,. \tag{3.1.9}$$

  where $Z(f)$ is the FT of $z(t)$.

- **Time support (TS):** The time support of $W_z(t,f)$ is limited by the duration of $z(t)$; that is, if $z(t)=0$ for $t<t_1$ and for $t>t_2$, then $W_z(t,f)=0$ for $t<t_1$ and for $t>t_2$.

- **Frequency support (FS):** The frequency support of $W_z(t,f)$ is limited by the bandwidth of $z(t)$; that is, if $\mathcal{F}\{z(t)\}=Z(f)=0$ for $f<f_1$ and for $f>f_2$, then $W_z(t,f)=0$ for $f<f_1$ and for $f>f_2$.

- **Convolution invariance:** The WVD of the time-convolution of two signals is the time-convolution of the WVDs of the two signals; that is, if

$$z_3(t) = z_1(t) \underset{t}{*} z_2(t)\ , \tag{3.1.10}$$

  then

$$W_{z_3}(t,f) = W_{z_1}(t,f) \underset{t}{*} W_{z_2}(t,f). \tag{3.1.11}$$

- **Modulation invariance:** The WVD of the frequency-convolution of two signals is the frequency-convolution of the WVDs of the two signals; that is, if

$$z_3(t) = z_1(t)\, z_2(t)\ , \tag{3.1.12}$$

  then

$$W_{z_3}(t,f) = W_{z_1}(t,f) \underset{f}{*} W_{z_2}(t,f). \tag{3.1.13}$$

- **Invertibility:** If $W_z(t,f)$ is the WVD of the signal $z(t)$, it may be shown [1, pp. 223–4] that

$$\int_{-\infty}^{\infty} W_z(t/2,f)\, e^{j2\pi f t} df = z(t)\, z^*(0). \tag{3.1.14}$$

  Putting $t=0$ in this result yields the magnitude, but not the phase, of $z(0)$. Hence a signal $z(t)$ may be recovered from its WVD *up to a phase factor.*

- **Inner-product invariance:** The WVD is a **unitary** transformation; that is, it preserves inner products:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_{z_1}(t,f)\, W_{z_2}(t,f)\, dt\, df = \left| \int_{-\infty}^{\infty} z_1(t)\, z_2^*(t)\, dt \right|^2 . \tag{3.1.15}$$

The above properties are not independent; for example, it is a trivial exercise to show that either of the marginals (TM or FM) implies the global energy condition.

A more comprehensive list of the WVD properties is given in [1]. These either follow directly from the definition, or are proven in [2].

A notable omission from the above list is **positivity**; the WVD can assume negative values, and indeed does so for almost every signal.

Applicability: Many properties of the WVD are desirable in applications; for example, realness is consistent with the notion of energy density in as much as energy is real, while convolution invariance and modulation invariance make the WVD partly compatible with linear filtering theory. Nevertheless, the WVD was not applied to real-life problems until the late 1970s, when it was implemented for the purposing of processing linear FM signals used in seismic prospecting [3].

3.1.2 Limitations of the WVD

Despite its many desirable properties, the WVD has some drawbacks. It may assume large negative values. Further, because it is **bilinear** in the signal rather than linear, it suffers from spurious features called **artifacts** or **cross-terms**, which appear midway between true signal components in the case of multicomponent signals as well as non-linear mono- and multicomponent FM signals [see Article 4.2].

3.1.2.1 Nonlinear Monocomponent FM Signals and Inner Artifacts

Consider a monocomponent signal $z(t) = a(t)\, e^{j\phi(t)}$. In the case of a *linear* FM signal, the WVD gives an accurate representation of the IF law (Fig. 1.1.3) because the CFD approximation to $\phi'(t)$ is exact, so that the signal kernel is a dechirped function of τ [see Eqs. (2.1.19) to (2.1.25)]. In the case of a *nonlinear* FM signal, the CFD approximation is *not* exact and the signal kernel (i.e. the IAF) is *not* dechirped. This results in the formation of **inner artifacts**, which arise from "within" a single component.

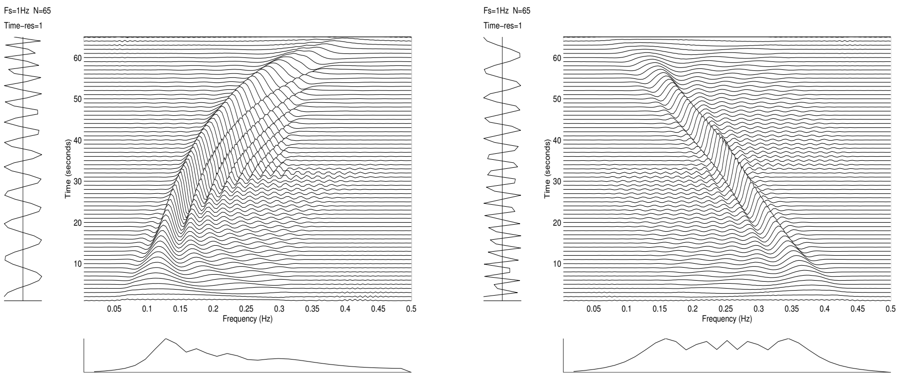
An example of a nonlinear FM signal is the **hyperbolic FM** signal with rectangular amplitude $a(t)$ and the phase given by

$$\phi(t) = \frac{2\pi f_0}{\alpha} \ln|1+\alpha t|. \tag{3.1.16}$$

The IF is

$$f_i(t) = \frac{\phi'(t)}{2\pi} = \frac{f_0}{1+\alpha t}. \tag{3.1.17}$$

For the time interval $0 \le t \le T$, the starting frequency is $f_0 = f_i(0)$. The finishing frequency is $f_{max} = f_i(T) = f_0/(1+\alpha T)$.



(a) (b)

Fig. 3.1.1: WVD of (a) hyperbolic FM signal with starting frequency 0.1 and finishing frequency 0.4 Hz; (b) linear FM signal with starting frequency 0.4 and finishing frequency 0.1 Hz. Both plots are for a duration of 65 samples (sampling rate 1 Hz).

Fig. 3.1.1(a) shows the WVD of a hyperbolic FM signal with duration 65 samples, $f_0 = 0.1$ and $f_{max} = 0.4$. While the crest of the WVD seems to be a reasonable approximation to the IF law, the energy concentration is poorer. The many spurious ridges are the inner artifacts. They alternate in sign as we move *normal* to the IF law in the (t, f) plane; this is a characteristic feature of inner artifacts.

For comparison, Fig. 3.1.1(b) shows the WVD of the linear FM signal with the same duration and the same frequency limits (with falling frequency). Note the superior energy concentration for the linear FM case, and the more attenuated artifacts distributed on both sides of the main ridge.

Artifacts caused by nonlinear FM laws can be reduced by windowing the IAF in τ before taking the FT, leading to the **windowed WVD**. This procedure, however, causes a loss of frequency resolution [4].

3.1.2.2 Multicomponent Signals and Outer Artifacts

If $z(t)$ is a multicomponent signal, the algebraic expansion of $K_z(t,\tau)$ contains cross-product terms which, when Fourier-transformed, give rise to spurious features in the WVD. These are the **outer artifacts** or **cross-terms**. To explain these, consider the signal

$$z(t) = z_1(t) + z_2(t) \tag{3.1.18}$$

where $z(t)$, $z_1(t)$ and $z_2(t)$ are analytic. Expanding the IAF, we obtain

$$K_z(t,\tau) = K_{z_1}(t,\tau) + K_{z_2}(t,\tau) + K_{z_1 z_2}(t,\tau) + K_{z_2 z_1}(t,\tau) \tag{3.1.19}$$

where $K_{z_1 z_2}(t,\tau)$ and $K_{z_2 z_1}(t,\tau)$ are the "signal cross-kernels" or instantaneous cross-correlation functions (e.g. $K_{z_1 z_2}(t,\tau) = z_1(t+\frac{\tau}{2})\, z_2^*(t-\frac{\tau}{2})$). Taking FTs of

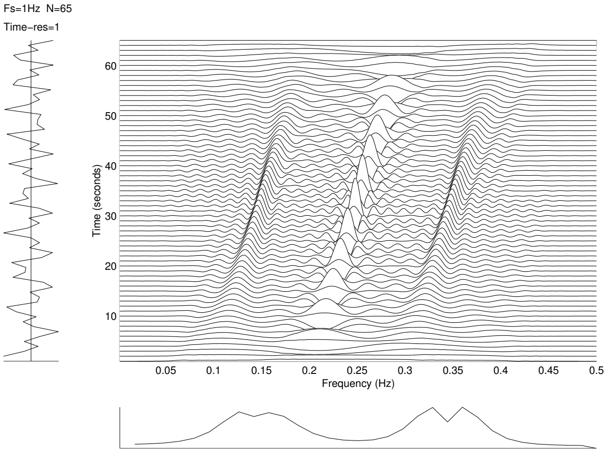


Fig. 3.1.2: The WVD of the sum of two linear FM signals with frequency ranges 0.1–0.2 and 0.3–0.4 Hz, unit amplitudes, and a duration of 65 samples (sampling rate 1 Hz).

Eq. (3.1.19) w.r.t. τ [using Eq. (2.1.18)], we find

$$W_z(t,f) = W_{z_1}(t,f) + W_{z_2}(t,f) + 2\mathrm{Re}\{W_{z_1 z_2}(t,f)\} \tag{3.1.20}$$

where $W_{z_1}(t,f)$ and $W_{z_2}(t,f)$ are the WVDs of $z_1(t)$ and $z_2(t)$, and $W_{z_1 z_2}(t,f)$ is the **cross-Wigner-Ville distribution** (XWVD) of $z_1(t)$ and $z_2(t)$, defined by

$$W_{z_1 z_2}(t,f) = \int_{-\infty}^{\infty} z_1(t+\tfrac{\tau}{2})\, z_2^*(t-\tfrac{\tau}{2})\, e^{-j2\pi f\tau} d\tau. \tag{3.1.21}$$

Thus the WVD of the sum of two signals is not just the sum of the signals' WVDs, but also of their XWVDs. If $z_1(t)$ and $z_2(t)$ are monocomponent signals, $W_{z_1}(t,f)$ and $W_{z_2}(t,f)$ are the auto-terms, while $2\mathrm{Re}\{W_{z_1 z_2}(t,f)\}$ is a cross-term.

Fig. 3.1.2 shows the WVD of the sum of two parallel linear FM signals. There seem to be three components rather than two; the "extra" component at the mean frequency of the expected components has a large oscillating amplitude, and occurs in a region of the (t,f) plane where we expect no energy at all. Fig. 3.1.3(a) shows the WVD of the sum of two FM signals crossing over in frequency. A large number of undulations appear in addition to the two main ridges. In both examples, the cross-terms alternate in sign as we move *parallel* to the expected features in the (t,f) plane; this is a characteristic feature of cross-terms.

3.1.2.3 Suppression of Cross-Terms

Cross-terms can make the WVD difficult to interpret, especially if the components are numerous or close to each other, and the more so in the presence of noise; cross-terms between signal components and noise exaggerate the effects of noise and

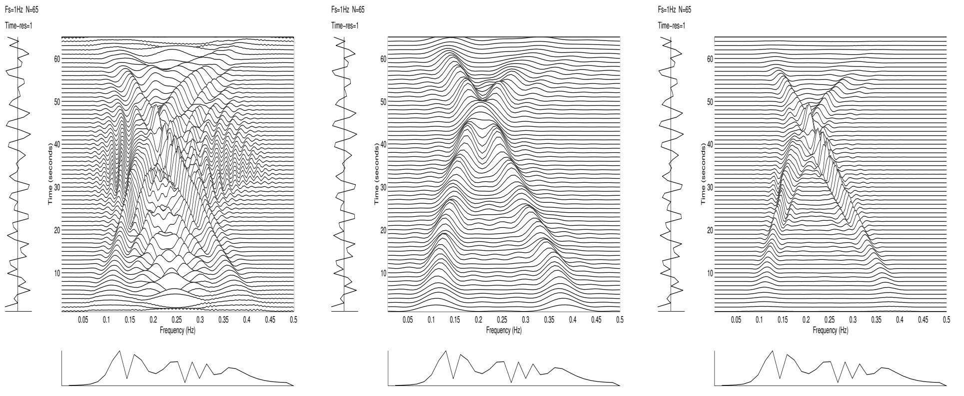


Fig. 3.1.3: Suppression of cross-terms in the sum of a rising hyperbolic FM signal (frequency range 0.1 to 0.4 Hz) and a falling linear FM signal (frequency range 0.4 to 0.1 Hz), with unit amplitudes and a duration of 65 samples (sampling rate 1 Hz): (a) WVD; (b) spectrogram with 21-point rectangular window; (c) masked WVD, being the product of (a) and (b).

cause rapid degradation of performance as the SNR decreases. For such reasons, cross-terms are often regarded as the fundamental limitation on the applicability of quadratic time-frequency methods, and the desire to suppress them has led to several approaches such as:

1. If we multiply the WVD by the spectrogram, we obtain the so-called **masked WVD** [5], which combines the cross-term suppression of the spectrogram with the high resolution of the WVD. To illustrate the effect, Fig. 3.1.3 shows the WVD, spectrogram and masked WVD of the sum of two crossed FM signals; observe that the masked WVD is "cleaner" than the WVD and has better resolution than the spectrogram.

2. Eq. (3.1.21) indicates that the XWVD of $z_1(t)$ and $z_2(t)$ is linear in $z_2(t)$. If $z_1(t)$ is a reference signal and $z_2(t)$ is the signal under analysis, there will be no cross-terms between components of $z_2(t)$. This observation inspired efforts to displace the WVD with the XWVD in relevant areas of application. A reference signal is readily available in optimal detection [6], sonar and radar [7], and seismic exploration [8]. In applications where reference signals are not available, a filtered version of the observed signal is used as a reference signal. The filtering procedure may use the IF as a critical feature of the signal; for example, the data-dependent TFDs defined in [9] use, as a reference signal, the signal component that maximizes the energy concentration in the TFD.

3. All quadratic TFDs in Chapter 2 can be identified by a time-lag kernel, including the spectrogram which suppresses cross-terms. By analyzing the properties of time-lag kernels, we can define and design TFDs that attenuate cross-terms.

A quadratic TFD in which the cross-terms are attenuated relative to the auto-terms is often called a **reduced-interference distribution** (**RID**).

This chapter concentrates on the RID approach for defining quadratic TFDs with the property of cross-terms suppression.

3.2 Formulations of Quadratic TFDs

3.2.1 Time-Lag Formulations and Other Domain Definitions

Given an analytic signal $z(t)$, the instantaneous autocorrelation function (IAF) was defined in Eq. (2.1.8) as

$$K_z(t,\tau) = z(t+\tfrac{\tau}{2})\, z^*(t-\tfrac{\tau}{2}) \tag{3.2.1}$$

The IAF $K_z(t,\tau)$ is a function of 2 time variables: the actual time t and the time lag τ. By taking the dual domain of t and τ in frequency, we obtain the frequency variables ν and f. This allows for four domains of representation:

$$\begin{gathered}(t,\tau)\\(t,f)\\(\nu,f)\\(\nu,\tau).\end{gathered}$$

Starting with the IAF, we define the Wigner-Ville distribution (WVD) by taking the FT $(\tau \to f)$:

$$W_z(t,f) = \underset{\tau\to f}{\mathcal{F}}\{K_z(t,\tau)\}. \tag{3.2.2}$$

The FT $(t \to \nu)$ of the WVD defines the spectral autocorrelation function (SAF) [Eq. (2.1.26)] as

$$k_z(\nu,f) = \underset{t\to \nu}{\mathcal{F}}\{W_z(t,f)\}. \tag{3.2.3}$$

The FT $(t \to \nu)$ of the IAF $K_z(t,\tau)$ equals the IFT $(\tau \leftarrow f)$ of the SAF $k_z(\nu,f)$, and defines the symmetrical **ambiguity function** (AF) as:

$$A_z(\nu,\tau) = \underset{t\to \nu}{\mathcal{F}}\{K_z(t,\tau)\} = \underset{\tau\leftarrow f}{\mathcal{F}^{-1}}\{k_z(\nu,f)\} \tag{3.2.4}$$

where $\mathcal{F}\{\ldots\}$ represents the forward (direct) Fourier transform, and $\mathcal{F}^{-1}\{\ldots\}$ its inverse.[1]

[1] Many authors define the symmetrical ambiguity function using an *inverse* transform from t to ν, with the result that the arrows in the "diamond diagrams" [e.g. Eq. (3.2.5)] point upwards instead of to the right (this affects only the lower-left and upper-right arrows). Here we choose our definitions so that transforms from a time variable (t or τ) to a frequency variable (ν or f) are always *forward*.

If we represent Fourier transformations by arrows labeled with the participating variables, Eqs. (3.2.2) to (3.2.4) may be combined into the single graphical equation

$$\begin{tikzcd}
 & W_z(t,f) \arrow[dr, "t", "\nu"'] & \\
K_z(t,\tau) \arrow[ur, "f", "\tau"'] \arrow[dr, "t", "\nu"'] & & k_z(\nu,f). \\
 & A_z(\nu,\tau) \arrow[ur, "f", "\tau"'] &
\end{tikzcd} \tag{3.2.5}$$

This single graphical representation of several equations is very useful in that it links several known methods by simple FT operations. The knowledge of FT properties then allows to use characteristics of a method in one domain and transfer them to another domain. We can so relate radar methods in the Doppler-lag domain to (t,f) methods and to cyclostationary methods.

Now let us represent similarly the TFD kernel in these four domains by taking various FTs of the **time-lag kernel** $G(t,\tau)$:

$$\begin{tikzcd}
 & \gamma(t,f) \arrow[dr, "t", "\nu"'] & \\
G(t,\tau) \arrow[ur, "f", "\tau"'] \arrow[dr, "t", "\nu"'] & & \mathcal{G}(\nu,f). \\
 & g(\nu,\tau) \arrow[ur, "f", "\tau"'] &
\end{tikzcd} \tag{3.2.6}$$

We refer to $g(\nu,\tau)$ as the **Doppler-lag kernel**, to $\mathcal{G}(\nu,f)$ as the **Doppler-frequency kernel**, and to $\gamma(t,f)$ as the **time-frequency kernel**.

3.2.2 Time-Frequency Formulation

We have defined the smoothed IAF as

$$R_z(t,\tau) = G(t,\tau) \underset{t}{*} K_z(t,\tau) \tag{3.2.7}$$

where $G(t,\tau)$ is the time-lag kernel. Following Eq. (2.7.1), we define a class of **quadratic TFDs** (which are smoothed WVDs) as

$$\rho_z(t,f) = \underset{\tau\to f}{\mathcal{F}}\{R_z(t,\tau)\}. \tag{3.2.8}$$

Substituting Eq. (3.2.7) into Eq. (3.2.8), writing out the convolution and the transform, and substituting from Eq. (3.2.1), we obtain

$$\rho_z(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(t-u,\tau)\, z(u+\tfrac{\tau}{2})\, z^*(u-\tfrac{\tau}{2})\, e^{-j2\pi f\tau} du\, d\tau \tag{3.2.9}$$

which defines the general class of quadratic TFDs in terms of the signal and the time-lag kernel. The class of TFDs of this form is called the **quadratic class**.[2]

Eq. (3.2.8) is included in the graphical equation

$$
\begin{tikzcd}
& \rho_z(t,f) \arrow[dr, "\nu", "t"'] & \\
R_z(t,\tau) \arrow[ur, "f", "\tau"'] \arrow[dr, "t", "\nu"'] & & r_z(\nu,f) \\
& \mathcal{A}_z(\nu,\tau) \arrow[ur, "f", "\tau"'] &
\end{tikzcd}
\tag{3.2.10}
$$

which assigns symbols to the various FTs of the smoothed IAF [11, p. 436]. By analogy with Eqs. (3.2.3) and (3.2.4), we call $r_z(\nu, f)$ the "generalized SAF" and $\mathcal{A}_z(\nu,\tau)$ the "generalized ambiguity function" (GAF)[3] or more precisely **filtered ambiguity function**; note the distinction between the normal A in Eq. (3.2.5) and the calligraphic $\mathcal{A}$ in Eq. (3.2.10).

Using Eq. (3.2.7) and the lower two arrows in Eq. (3.2.10), together with the convolution properties of the FT, we obtain in sequence

$$\mathcal{A}_z(\nu,\tau) = g(\nu,\tau)\, A_z(\nu,\tau) \tag{3.2.11}$$

$$r_z(\nu,f) = \mathcal{G}(\nu,f) \underset{f}{*} k_z(\nu,f) \tag{3.2.12}$$

Then, using Eq. (3.2.7) and the upper left arrow in Eq. (3.2.10), together with the convolution properties, we obtain

$$\rho_z(t,f) = \gamma(t,f) \underset{(t,f)}{**} W_z(t,f) \tag{3.2.13}$$

where the double asterisk denotes double convolution in t and f (cf. [11], pp. 437, 475).

Eq. (3.2.13) defines the quadratic class of TFDs in terms of the WVD and the time-frequency kernel. For this reason we regard the WVD as the "basic" or "prototype" quadratic TFD, and all other quadratic TFDs as filtered versions thereof.

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[2] The **quadratic class** as defined here satisfies the time-shift- and frequency-shift-invariance properties. Most current authors seem to equate "**Cohen's class**" with the quadratic class. In addition, some authors apply the term "quadratic time-frequency representation" or "QTFR" to all time-frequency representations that are quadratic in the signal (whether they have the form of Eq. (3.2.9) or not) so that "Cohen's class" becomes a subset of the "QTFRs". However, the class originally defined by Cohen in a quantum-mechanical context [10] differs from the quadratic class [Eq. (3.2.9)] in that the marginal conditions must be satisfied, while the kernel may depend on the signal (in which case the TFD is *not* quadratic in the signal). We shall see (e.g. in Table 3.3.3) that many useful quadratic TFDs do not satisfy the marginals, so that they are not members of "Cohen's class" as originally defined.

[3] The term **generalized ambiguity function** is also used with a different meaning in connection with polynomial TFDs; see Article 5.5.

Eq. (3.2.13) also suggests a method of suppressing cross-terms in the WVD. Because inner artifacts alternate as we move in the frequency direction, they can be attenuated by choosing a $\gamma(t, f)$ with a sufficient spread in the frequency direction. Similarly, because cross-terms (outer artifacts) alternate as we move in the time direction, they can be attenuated by choosing a $\gamma(t, f)$ with a sufficient spread in the time direction.

Eqs. (3.2.10) and (3.2.11) show how one quadratic TFD can be transformed to another by a linear transformation.

3.2.3 Doppler-Lag Formulation and TFD Design

Substituting for $K_z(t, \tau)$ in Eq. (3.2.4) and writing out the transform, we obtain

$$A_z(\nu, \tau) = \int_{-\infty}^{\infty} z(t + \tfrac{\tau}{2})\, z^*(t - \tfrac{\tau}{2})\, e^{-j2\pi\nu t} dt \tag{3.2.14}$$

$$= \int_{-\infty}^{\infty} z(t + \tfrac{\tau}{2})\, \left[z(t - \tfrac{\tau}{2})\, e^{j2\pi\nu t}\right]^* dt. \tag{3.2.15}$$

The expression in square brackets can be obtained by delaying $z(t + \frac{\tau}{2})$ in time by τ and shifting it in frequency by ν, indicating that $A_z(\nu, \tau)$ is the correlation of the signal with a time-delayed and frequency-shifted version of itself. This correlation is well known in radar theory as the Sussman ambiguity function [12]; the name "ambiguity" arises from the equivalence between time-shifting and frequency-shifting for linear FM signals, which are frequently used in radar. Hence the Doppler-lag (ν, τ) domain is also called the **ambiguity domain**.

Eq. (3.2.10) indicates that the quadratic TFD is a 2D FT (half inverse, half forward) of the filtered ambiguity function. Writing out the transforms gives

$$\rho_z(t, f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\nu, \tau)\, A_z(\nu, \tau)\, e^{j2\pi(\nu t - f\tau)} d\nu\, d\tau. \tag{3.2.16}$$

Then writing the dummy u for t in Eq. (3.2.14) and substituting the result into Eq. (3.2.16) gives

$$\rho_z(t, f) = \iiint g(\nu, \tau)\, z(u + \tfrac{\tau}{2})\, z^*(u - \tfrac{\tau}{2})\, e^{j2\pi(\nu t - \nu u - f\tau)} du\, d\nu\, d\tau \tag{3.2.17}$$

where the integrals are from $-\infty$ to ∞; this defines the quadratic TFD in terms of the Doppler-lag kernel $g(\nu, \tau)$.

From the lower left side of Eq. (3.2.6), the relationship between the time-lag kernel and the Doppler-lag kernel is

$$g(\nu, \tau) = \underset{t \to \nu}{\mathcal{F}} \{G(t, \tau)\}. \tag{3.2.18}$$

Using this equation, the time-lag kernels determined in Section 2.7 may be converted to Doppler-lag form. For example, using Eq. (2.7.7), we find that the Doppler-lag kernel for the WVD is $g(\nu, \tau) = \underset{t \to \nu}{\mathcal{F}} \{\delta(t)\} = 1$.

Eq. (3.2.13) and its 2D FT Eq. (3.2.16) express that a quadratic TFD can be designed using basic filter design principles. As in 1D filter design, the filter specifications are best given in the domain where the filtering operation is expressed as a multiplication as in Eq. (3.2.11), rather than in the dual domain with a convolution as in Eq. (3.2.13).

If the Doppler-lag kernel $g(\nu, \tau)$ has the separable form $G_1(\nu)\, g_2(\tau)$, multiplication by this kernel may include the combined effect of time windowing and frequency windowing (see the discussion of Eq. (2.1.40)).

We can use the same process to define a filter which attenuates cross-terms in quadratic TFDs. The cross-terms in the (t, f) domain tend to be highly oscillatory, so that the corresponding terms in the dual (ν, τ) domain tend to be far from the origin (high-pass). The auto-terms in the (t, f) domain tend to be smooth and well delineated, so that the corresponding terms in the dual (ν, τ) domain are concentrated about the origin or "pass through" the origin (low-pass). This behavior is well known in the field of radar [13, 14]. Hence the cross-terms in the ambiguity domain can be "filtered out" by selecting a kernel filter[4] $g(\nu, \tau)$ that deemphasizes information far from the origin in the Doppler-lag domain.

Various authors [10, 15, 16] have shown that other desirable TFD properties are equivalent to constraints on the kernel filter, and we shall see that most of these constraints are conveniently expressed in the Doppler-lag domain. For all the above reasons, *the design of the TFD kernel filter is usually performed in the Doppler-lag domain.* Often, the resulting kernel is then described in the time-lag domain for ease of TFD implementation, as given by Eq. (3.2.9).

3.2.4 Doppler-Frequency Formulation

From Eqs. (3.2.10) and (3.2.12) we have

$$\rho_z(t, f) = \underset{t \leftarrow \nu}{\mathcal{F}^{-1}} \left\{ \mathcal{G}(\nu, f) \underset{f}{*} k_z(\nu, f) \right\} \tag{3.2.19}$$

or, writing out the transform and convolution,

$$\rho_z(t, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{G}(\nu, f - \eta)\, k_z(\nu, \eta)\, e^{j2\pi\nu t} d\eta\, d\nu \tag{3.2.20}$$

Writing η for f in Eq. (2.1.31) and substituting the result into Eq. (3.2.20), we obtain

$$\rho_z(t, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{G}(\nu, f - \eta)\, Z(\eta + \tfrac{\nu}{2})\, Z^*(\eta - \tfrac{\nu}{2})\, e^{j2\pi\nu t} d\eta\, d\nu. \tag{3.2.21}$$

This defines quadratic TFDs in terms of the Doppler-frequency kernel $\mathcal{G}(\nu, f)$ and the signal spectrum. The interest of this formulation is that TFDs of narrow-band signals expressed by their spectra may be more efficiently computed in this form.

[4] The term "kernel filter" is used to reinforce the idea that designing quadratic TFDs essentially reduces to filter design with specific constraints.

3.2.5 Examples of Simple TFD Formulations

3.2.5.1 Separable Kernels

A simple way to design kernel filters for quadratic TFDs is to consider the special case of a separable kernel (see Article 5.7):

$$g(\nu,\tau) = G_1(\nu)\, g_2(\tau). \tag{3.2.22}$$

If we let

$$G_1(\nu) = \underset{t\to\nu}{\mathcal{F}}\{g_1(t)\} \tag{3.2.23}$$

and

$$G_2(f) = \underset{\tau\to f}{\mathcal{F}}\{g_2(\tau)\}, \tag{3.2.24}$$

then Eq. (3.2.11) becomes

$$\mathcal{A}_z(\nu,\tau) = G_1(\nu)\, g_2(\tau)\, A_z(\nu,\tau). \tag{3.2.25}$$

Then, using Eqs.(3.2.10), (3.2.23), and(3.2.24), we obtain

$$\rho_z(t,f) = g_1(t) \underset{t}{*} W_z(t,f) \underset{f}{*} G_2(f). \tag{3.2.26}$$

Eq. (3.2.25) shows that the design of the kernel filter $G_1(\nu)\, g_2(\tau)$ is greatly simplified as the 2D filtering operation is replaced by two successive 1D filtering operations. Equivalently, in Eq. (3.2.26), the two convolutions can be evaluated in either order, indicating that the Doppler-dependent and lag-dependent factors in the separable kernel $g(\nu,\tau)$ lead to separate convolutions in time and frequency, respectively.

A separable kernel is not to be confused with a **product kernel**, which is a function of the product $\nu\tau$, e.g.

$$g(\nu,\tau) = g_3(\nu\tau). \tag{3.2.27}$$

3.2.5.2 Doppler-Independent Kernels

A **Doppler-independent** (DI) kernel is a special case of a separable kernel obtained by putting

$$G_1(\nu) = 1 \tag{3.2.28}$$

in Eqs. (3.2.22) and (3.2.23), which then become

$$g(\nu,\tau) = g_2(\tau) \tag{3.2.29}$$

$$g_1(t) = \delta(t). \tag{3.2.30}$$

Making these substitutions in Eqs. (3.2.25) and (3.2.26), we obtain

$$\mathcal{A}_z(\nu,\tau) = g_2(\tau)\, A_z(\nu,\tau) \tag{3.2.31}$$

$$\rho_z(t,f) = G_2(f) \underset{f}{*} W_z(t,f). \tag{3.2.32}$$

The last result shows that a DI kernel is obtained by applying only 1D filtering and causes smearing of the WVD in the frequency direction *only*.

3.2.5.3 Lag-Independent Kernels

A **lag-independent** (LI) kernel is another special case of the separable kernel, obtained by putting

$$g_2(\tau) = 1 \tag{3.2.33}$$

in Eqs. (3.2.22) and (3.2.24), which then become

$$g(\nu,\tau) = G_1(\nu) \tag{3.2.34}$$

$$G_2(f) = \delta(f). \tag{3.2.35}$$

Making these substitutions in Eqs. (3.2.25) and (3.2.26), we obtain

$$\mathcal{A}_z(\nu,\tau) = G_1(\nu)\, A_z(\nu,\tau) \tag{3.2.36}$$

$$\rho_z(t,f) = g_1(t) \underset{t}{*} W_z(t,f). \tag{3.2.37}$$

The last result shows that an LI kernel is obtained by applying only 1D filtering and causes smearing of the WVD in the time direction *only*.

The kernel of the WVD is $g(\nu,\tau) = 1$, which is both Doppler-independent and lag-independent. The windowed WVD kernel, however, is Doppler-independent.

Article 5.7 provides an in-depth treatment of such separable kernels, including examples of quadratic TFDs obtained by this simple kernel filter design procedure.

3.3 Properties of Quadratic TFDs

3.3.1 Desirable Properties

Some quadratic TFDs verify desirable properties that are not shared by the WVD, and vice versa. Later we shall relate the properties of a TFD to the constraints of its kernel, and tabulate properties for selected TFDs. But first we discuss some properties that have been promoted [17–19] as fundamental for a wide range of applications.

1. **Concentration of local energy:** The energy in a certain region R in the (t,f) plane, denoted by E_{z_R}, is given by the integral of the TFD over the region R; for example, if R is the region within a time interval Δt and a frequency band Δf, the energy within R is

$$E_{z_R} = \int_{\Delta t}\int_{\Delta f} \rho_z(t,f)\, df\, dt. \tag{3.3.1}$$

2. **IF/TD visualization:** The TFD of a monocomponent signal directly depicts the instantaneous frequency $f_i(t)$ and time delay $\tau_\mathrm{d}(f)$ as a range of peaks along the curve representing the FM law in the (t,f) plane. That is, if $z(t)$ is a monocomponent signal and $\rho_z(t,f)$ is its TFD, then

$$\max_f \rho_z(t,f) = f_i(t). \tag{3.3.2}$$

3. **Reduced interference (RI):** The TFD attenuates or suppresses inner and outer artifacts (cross-terms) relative to the signal components (auto-terms).

Another condition considered by some authors is **positivity**, or more precisely **non-negativity (NN)**, defined as:

$$\rho_z(t,f) \geq 0 \;\; \forall \;\; z, t, f. \tag{3.3.3}$$

Among quadratic TFDs, only *sums of spectrograms* possess NN. This means that the Doppler-lag kernel of a "positive" TFD is a sum of ambiguity functions, which makes the NN property incompatible with both the IF property (see Table 3.3.1) and IF visualization [20]. Hence NN is usually considered non-essential because its cost is excessive.

A subclass $\mathcal{P}$ of quadratic TFDs comprises those TFDs which satisfy the realness, time marginal, frequency marginal, instantaneous frequency, time support and frequency support properties. Researchers have shown much interest in TFDs of this class, and especially RIDs of this class. To some extent the design of RIDs is the art of "improving" on the spectrogram by

- sacrificing NN,
- improving resolution, and
- retaining sufficient reduced-interference ability for the application.

The result is usually a compromise between the spectrogram and the WVD, involving a time-frequency kernel filter less extensive than that of the spectrogram. The compromise may involve sacrificing one or both of the marginal properties (time or frequency). Although the marginals are critical in the field in quantum physics [10], they seem to be less important in signal processing. As evidence of this we may cite the following:

- The spectrogram does not satisfy the marginals and yet has always been regarded as a very useful TFD.
- Attempts to improve on the spectrogram are most often motivated by the need for higher resolution (see e.g. [21]) rather than any desire to satisfy the marginals.
- If the marginal conditions do not hold, it is still possible that the integral forms thereof [Eqs. (2.6.6) and (2.6.7)] are approximately true over sufficiently broad time intervals or frequency bands.

For these reasons it is suggested that $\mathcal{P}$ is not necessarily appropriate for signal-processing applications, and an alternative class $\mathcal{P}'$, which comprises reduced-interference distributions satisfying the realness, global and local energy, IF visualization and components resolution properties, would be more relevant. Note that the properties of class $\mathcal{P}'$ are the same as the properties listed at the end of Section 1.1.5.

3.3.2 TFD Properties & Equivalent Kernel Constraints

Using some properties listed in [22], the relevant kernel constraints were adapted for separable, DI and LI kernels. The results are collected in Table 3.3.1.

The proofs of these kernel constraints use Eq. (3.2.13), which states that the quadratic TFD is the WVD convolved in t and f with the time-frequency kernel $\gamma(t, f)$:

$$\rho_z(t, f) = \gamma(t, f) \underset{(t,f)}{**} W_z(t, f).$$

Using this relation, we can find *sufficient* conditions under which certain properties of the WVD carry over to $\rho_z(t, f)$. For example:

- Realness holds if $\gamma(t, f)$ is real.
- Because 2D convolution is shift-invariant, time- and frequency-shift invariance hold for any fixed $\gamma(t, f)$.
- Time support and time extent hold for a DI kernel, which does not redistribute the WVD in time, whereas frequency support and frequency extent hold for an LI kernel, which does not redistribute the WVD in frequency.
- The IF moment property holds for a DI kernel if $G_2(f)$ has a first moment (mean frequency) of zero, so that convolution w.r.t. f does not change the first moment of the WVD w.r.t. f. Similarly, the GD property holds for an LI kernel if $g_1(t)$ has a first moment (mean time) of zero, so that convolution w.r.t. t does not change the first moment of the WVD w.r.t. t.

Some further proofs of kernel constraints are given by Cohen [23].

3.3.3 Examples of TFDs with Specific Properties

For the TFDs defined so far, Table 3.3.2 lists their kernels in various domains, and Table 3.3.3 shows their properties.

Note: A time-lag kernel satisfying the time-support constraint is described as a **butterfly function** [11] or **cone-shaped kernel** [24]; that is, the nonzero values of the kernel are confined to the interior of a two-dimensional cone in the (t, τ) plane (see the entry in the "General" column of Table 3.3.1). The Born-Jordan and ZAM distributions have kernels of this type.

Inspection of Table 3.3.1 confirms that the WVD satisfies all of the listed properties except reduced interference (RI). The reduced-interference capabilities of separable kernels warrant special attention. Because the inner artifacts alternate as we move normal to the components, they also alternate in the frequency direction and can be suppressed by convolution with a sufficiently long $G_2(f)$ [see Eq. (3.2.32)], which corresponds to a sufficiently short $g_2(\tau)$. This is possible for a DI kernel but not an LI kernel. Because the cross-terms (outer artifacts) alternate as we move parallel to the components, they also alternate in the time direction and can be suppressed by convolution with a sufficiently long $g_1(t)$ [see Eq. (3.2.37)], which corresponds to a sufficiently short $G_1(\nu)$. This is possible for an LI kernel but not a

Table 3.3.1: TFD properties and associated constraints on the Doppler-lag kernel for general, separable, Doppler-independent and lag-independent kernels. The asterisk in "WVD only*" means that the WVD may be multiplied by a scale factor. **Abbreviations for properties:** NN = non-negativity; RE = realness; TI = time-shift invariance (or time covariance); FI = frequency-shift invariance (or frequency covariance); TM = time marginal; FM = frequency marginal; IF = instantaneous frequency; TD = time delay; TS = time support; FS = frequency support; RI = RID capability.

Property	KERNEL CONSTRAINTS			
	General —	Separable $g(\nu,\tau) = G_1(\nu)\, g_2(\tau)$	DI $g(\nu,\tau) = g_2(\tau)$	LI $g(\nu,\tau) = G_1(\nu)$
NN:	$G(t,\tau)$ is a sum of IAFs.	$g_1(t)\, g_2(\tau)$ is a sum of IAFs.	Never	Never
RE:	$g(\nu,\tau) = g^*(-\nu,-\tau)$.	$G_1(\nu)\, g_2(\tau) = G_1^*(-\nu)\, g_2^*(-\tau)$.	$G_2(f)$ is real.	$g_1(t)$ is real.
TI:	$g(\nu,\tau)$ does not depend on t.	$G_1(\nu)\, g_2(\tau)$ does not depend on t.	$g_2(\tau)$ does not depend on t.	$G_1(\nu)$ does not depend on t.
FI:	$g(\nu,\tau)$ does not depend on f.	$G_1(\nu)\, g_2(\tau)$ does not depend on f.	$g_2(\tau)$ does not depend on f.	$G_1(\nu)$ does not depend on f.
TM:	$g(\nu,0) = 1 \;\; \forall\nu$	$G_1(\nu)\, g_2(0) = 1 \;\; \forall\nu$	$g_2(0) = 1$	WVD only
FM:	$g(0,\tau) = 1 \;\; \forall\tau$	$G_1(0)\, g_2(\tau) = 1 \;\; \forall\tau$	WVD only	$G_1(0) = 1$
IF:	$g(\nu,0) = \text{const.}$ $\frac{\partial g}{\partial \tau}\big\rvert_{\tau=0} = 0 \;\; \forall\nu.$	$G_1(\nu)\, g_2(0) = \text{const.}$ $g_2'(0) = 0.$	$g_2'(0) = 0$	WVD only*
TD:	$g(0,\tau) = \text{const.}$ $\frac{\partial g}{\partial \nu}\big\rvert_{\nu=0} = 0 \;\; \forall\tau.$	$G_1(0)\, g_2(\tau) = \text{const.}$ $G_1'(0) = 0.$	WVD only*	$G_1'(0) = 0$
TS:	$G(t,\tau) = 0$ if $\lvert\tau\rvert < 2\lvert t\rvert$.	DI only	Always	WVD only*
FS:	$\mathcal{G}(f,\nu) = 0$ if $\lvert\nu\rvert < 2\lvert f\rvert$.	LI only	WVD only*	Always
RI:	Unrestricted	Unrestricted	Inner x-terms	Outer x-terms

DI kernel. A general separable kernel with sufficiently short $G_1(\nu)$ and $g_2(\tau)$ can therefore attenuate both kinds of artifacts; a TFD with such a kernel is shown in Fig. 2.7.1 part (l) for a linear FM signal.

Properties of separable kernels and their special cases, such as the B-distribution and Modified B-distribution, are discussed further and illustrated on examples in Article 5.7. Design of RIDs is discussed more generally in Article 5.2.

Table 3.3.2: Kernels of selected TFDs in the time-lag, Doppler-lag, and (where possible) Doppler-frequency domains. The window $w(t)$ is assumed to be real and even. Its FT and AF are $W(f)$ and $A_w(\nu,\tau)$, respectively. The prefix "w-" means "windowed".

Distribution	KERNEL $G(t,\tau)$	KERNEL $g(\nu,\tau)$	KERNEL $\mathcal{G}(\nu,f)$
Wigner-Ville	$\delta(t)$	1	$\delta(f)$
Levin	$\frac{1}{2}\left[\delta(t+\frac{\tau}{2})+\delta(t-\frac{\tau}{2})\right]$	$\cos(\pi\nu\tau)$	$\frac{1}{2}\left[\delta(f+\frac{\nu}{2})+\delta(f-\frac{\nu}{2})\right]$
Born-Jordan	$\frac{1}{\lvert 2\alpha\tau\rvert}\operatorname{rect}\frac{t}{2\alpha\tau}$	$\operatorname{sinc}(2\alpha\nu\tau)$	$\frac{1}{\lvert 2\alpha\nu\rvert}\operatorname{rect}\frac{f}{2\alpha\nu}$
Modified B	$\frac{\cosh^{-2\beta}t}{\int_{-\infty}^{\infty}\cosh^{-2\beta}\xi\,d\xi}$	$\frac{\lvert\Gamma(\beta+j\pi\nu)\rvert^2}{\Gamma^2(\beta)}$	$\frac{\lvert\Gamma(\beta+j\pi\nu)\rvert^2}{\Gamma^2(\beta)}\,\delta(f)$
w-WVD	$\delta(t)\,w(\tau)$	$w(\tau)$	$W(f)$
w-Levin	$\frac{w(\tau)}{2}\left[\delta(t+\frac{\tau}{2})+\delta(t-\frac{\tau}{2})\right]$	$w(\tau)\cos(\pi\nu\tau)$	$\frac{1}{2}\left[W(f+\frac{\nu}{2})+W(f-\frac{\nu}{2})\right]$
ZAM	$w(\tau)\operatorname{rect}\frac{t}{2\tau/a}$	$w(\tau)\frac{a}{2\lvert\tau\rvert}\operatorname{sinc}\frac{2\nu\tau}{a}$	
Rihaczek	$\delta(t-\frac{\tau}{2})$	$e^{-j\pi\nu\tau}$	$\delta(f+\frac{\nu}{2})$
w-Rihaczek	$w(\tau)\,\delta(t-\frac{\tau}{2})$	$w(\tau)\,e^{-j\pi\nu\tau}$	$W(f+\frac{\nu}{2})$
Page	$\delta(t-\lvert\frac{\tau}{2}\rvert)$	$e^{-j\pi\nu\lvert\tau\rvert}$	$\frac{1}{2}\left[\delta(f+\frac{\nu}{2})+\delta(f-\frac{\nu}{2})\right] + j\nu/[2\pi(f^2-\nu^2/4)]$
Choi-Williams	$\frac{\sqrt{\pi\sigma}}{\lvert\tau\rvert}e^{-\pi^2\sigma t^2/\tau^2}$	$e^{-\nu^2\tau^2/\sigma}$	$\frac{\sqrt{\pi\sigma}}{\lvert\nu\rvert}e^{-\pi^2\sigma f^2/\nu^2}$
B	$\lvert\tau\rvert^{\beta}\cosh^{-2\beta}t$	$\frac{\lvert\tau\rvert^{\beta}\,\lvert\Gamma(\beta+j\pi\nu)\rvert^2}{2^{1-2\beta}\,\Gamma(2\beta)}$	
Spectrogram	$w(t+\frac{\tau}{2})\,w(t-\frac{\tau}{2})$	$A_w(\nu,\tau)$	$W(f+\frac{\nu}{2})\,W(f-\frac{\nu}{2})$

3.4 Summary, Discussion and Conclusions

This chapter ends the tutorial introduction to TFSAP constituted by Part I of this book. In essence, the main message and findings of this chapter are as follows. For a monocomponent linear FM signal, the WVD is optimal for energy concentration about the IF and for unbiased estimation of the IF. If a signal has nonlinear frequency modulation and/or multiple components, the WVD suffers from inner artifacts and/or outer artifacts (cross-terms), respectively; in either case, some form of reduced interference quadratic TFD (RID) is to be preferred over the WVD. The design of RIDs is best undertaken by designing the desired kernel filter in the ambiguity domain, and using Fourier transforms to see the effects in the time-lag and time-frequency domains. To be a useful tool for practical applications, quadratic TFDs are expected to be real, to satisfy the global and local energy requirements, and to resolve signal components while reflecting the components' IF laws through the peaks of their dominant ridges in the (t,f) plane. Several RIDs were designed

Table 3.3.3: Properties of the TFDs whose kernels are listed in Table 3.3.2. The window $w(t)$ is assumed to be real and even. An exclamation (!) means that the property is always *satisfied*, while an asterisk (*) means that the property is satisfied subject to normalization of the window (for TM) or the value of the parameter (for TS and FS). Comments on the kernel type are in parentheses (thus).

	PROPERTY						
Distribution	RE	TM	FM	IF	TD	TS	FS
Wigner-Ville	!	!	!	!	!	!	!
Levin (product kernel)	!	!	!	!	!	!	!
Born-Jordan (product kernel)	!	!	!	!	!	*	*
Modified B (LI kernel)	!		!		!		!
w-WVD (DI kernel)	!	*		!		!	
w-Levin	!	*		!		!	
ZAM	!	*		!		*	
Rihaczek (product kernel)		!	!			!	!
w-Rihaczek		*				!	
Page	!	!	!	!		!	
Choi-Williams (product kernel)	!	!	!	!	!		
B (separable kernel)	!						
Spectrogram	!						

using simple separable kernels, demonstrating the procedure to construct quadratic TFDs that meet the above requirements. Special-purpose quadratic TFDs can be easily designed to meet the specifications of particular applications (as we shall see in several articles in the following chapters). In general, however, the WVD and various RIDs are the most useful TFDs; the spectrogram, which has been widely used, is at best subsumed by quadratic RIDs (of which it is a special case), and at worst made obsolete by them.

The remaining four Parts of the book elaborate on these issues, discuss advanced design methods for TFDs, and present a wide selection of methodologies, algorithms, and applications that demonstrate how to use time-frequency methods in practice. In particular, questions and issues such as how to select a TFD for a particular application, how to implement it and how best to apply it are covered. These chapters constituting the remaining four Parts include a number of Articles which tend to cover more advanced and detailed material, complementing and supplementing the tutorial introduction of Chapter 1.

Part II of the book gives more details on some fundamental topics of TFSAP

such as TFD design and signal analysis in the (t, f) plane. It includes two chapters (4 and 5). Chapter 4 presents some advanced concepts for TF signal analysis, TF signal processing and TF system analysis. Chapter 5 presents a number of methods for designing TFDs, including a treatment of the ambiguity function.

Part III of the book describes specialized techniques used in implementation, measurement and enhancement of TFDs. It includes three chapters (6, 7 and 8). Chapter 6 deals with the implementation and realization of TFDs; in particular, the formulation of discrete-time quadratic TFDs is presented and computation issues are described. Chapter 7 presents quality measures for TFDs and methods for performance enhancement. Chapter 8 describes methods and algorithms for multi-sensor and time-space processing used in applications such as sonar and telecommunications.

Part IV presents the key statistical techniques for TFSAP of random signals. It includes four chapters (9 to 12). Chapter 9 presents time-frequency methods for random processes and noise analysis; Chapter 10 describes methods for instantaneous frequency estimation; Chapter 11 deals with the field of time-frequency synthesis and filtering, including time-varying filter design; and Chapter 12 presents time-frequency methods for signal detection, classification and estimation.

Part V describes a representative selection of TFSAP applications encompassing a wide range of fields and industries. It includes three chapters (13 to 15). Chapter 13 presents time-frequency applications in telecommunications; Chapter 14 describes time-frequency methods in radar, sonar and acoustics; and chapter 15 details a number of time-frequency methods for diagnosis and monitoring used in a wide range of diverse applications.

A detailed description of the contents of each chapter can be found in the table of contents of the book and in the introduction to each chapter. For a more detailed search of a topic needed, e.g. for the explanation of an advanced concept, the reader is referred to the detailed index at the end of the book.

For various reasons such as space limitations and publication delays, a number of recent references could not be included in the relevant chapters. They are sufficiently important to be listed here for the sake of completeness. They include new theoretical developments on a range of topics discussed elsewhere in this book such as TFDs with complex argument [25], rotated (t, f) kernel filters [26], Gabor analysis [27], detection [28], spectrogram segmentation [29], time-frequency plane decomposition in atoms [30], the issue of positivity [31, 32], IF estimation [33, 34], feature extraction [35], an illustration of the concept of BT product [36], polyspectra [37] and Volterra series [38].

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